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INFORMATION AND OPTIMAL INVESTMENT IN DEFAULTABLE ASSETS

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ABSTRACT. We study optimal investment in assets subject to risk of default for investors that rely on different levels of information. The price dynamics can include noises both from a Wiener process and a Poisson random measure with infinite activity. The default events are modeled via doubly stochastic Poisson processes in line with large part of the literature in credit risk. In order to deal with both cases of inside and partial information we consider the framework of the anticipating calculus of forward integration. This does not require the assumptions typical of the framework of enlargement of filtrations. We then solve the optimization problem for maximum expected utility at terminal time for a large class of utility functions. Various examples are provided.

1. INTRODUCTION: THE MODEL, THE OPTIMIZATION PROBLEM, THE STREAMS OF INFORMATION

Occasionally, we observe that bankruptcy or related events unexpectedly wipe out shareholder values. Inspired by default risk literature, we consider a model for stocks where there is a varying risk of instantaneous loss in stock value.

Our model market consists of a bond S_0 serving as numéraire with dynamics:

$$(1.1) \quad \begin{aligned} dS_0(t) &= S_0(t)\rho(t)dt, \\ S_0(0) &= 1 \end{aligned}$$

and a defaultable stock S_1 with price dynamics:

$$(1.2) \quad \begin{aligned} dS_1(t) &= S_1(t-)\mathbf{1}_{\{\tau > t\}} \left(\mu(t)dt + \sigma(t)dW(t) \right. \\ &\quad \left. + \int_{\mathbb{R}_0} \theta(t, z)d\tilde{N}(dt, dz) + \kappa(t)dH(t) \right), \\ S_1(0) &> 0. \end{aligned}$$

Here $W(t)$, $t \geq 0$, is a standard Wiener process and $\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$ is the compensated version of a Poisson random measure $N(dt, dz)$, $t \geq 0$, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, independent of W , with $\mathbb{E}[N(dt, dz)] = \nu(dz)dt$. The Borel measure $\nu(dz)$ on \mathbb{R}_0 is σ -finite and satisfies

$$\int_{\mathbb{R}_0} 1 \wedge z^2 \nu(dz) < \infty.$$

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The process $H(t)$, $t \geq 0$, is a doubly stochastic Poisson process that models the occurrence of default events. The use of doubly stochastic Poisson processes, also known as Cox processes, was introduced as a template for default intensities by Lando in [21], while the intensity based approach itself started with [19]. See also e.g. [4], [5], and [16]. In short the doubly stochastic Poisson process $H(t)$, $t \geq 0$, is a counting process with

$$(1.3) \quad \mathbb{P}(H(t) = k) = \mathbb{E}\left[\frac{e^{-\Lambda_t}(\Lambda_t)^k}{k!}\right], \quad k = 0, 1, \dots,$$

where Λ_t , $t \geq 0$, is stochastic. To avoid ambiguity we always choose to work with the càdlàg version of $H(t)$, $t \geq 0$.

Following the standard literature in this area, we consider Λ_t , $t \geq 0$, to be of the form:

$$(1.4) \quad \Lambda_t = \int_0^t \lambda(s) ds,$$

where λ is a non-negative stochastic process which is continuous in probability and such that $\lambda \in L^1(dt \times \mathbb{P})$. See for instance [17] for details. In applications to default risk, the jumps of H represent some form of default event. Typically, the interest is focused on the first n jumps. Here we will restrict ourselves to the first jump τ signifying default:

$$(1.5) \quad \tau = \inf\{t : H(t) > 0\}.$$

Note that the probabilities of default are given by:

$$\mathbb{P}(\tau > s | \tau > t) = \mathbf{1}_{\{\tau > t\}} \mathbb{E}\left[e^{-\int_t^s \lambda(u) du}\right], \quad s > t.$$

The integration with respect to H in (1.2) and in the sequel is meant ω -wise.

The goal of an investor in this market is to optimize his investments in S_1 depending on his knowledge of Λ , i.e. depending on his knowledge of the default risk, together with his knowledge of the coefficients μ, σ, θ , and κ . On the complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we single out different streams of information, modeled by filtrations, playing different roles in this study:

- $\mathbb{H}^\tau := \{\mathcal{H}_t^\tau, t \geq 0\}$ where $\mathcal{H}_t^\tau = \sigma\{H(s), s \leq t\}$
- $\mathbb{H}^\Lambda := \{\mathcal{H}_t^\Lambda, t \geq 0\}$ where $\mathcal{H}_t^\Lambda = \sigma\{\Lambda(s), s \leq t\}$
- $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ where $\mathcal{F}_t = \sigma\{W(s), N((s, t], B), s \leq t, B \in \mathcal{B}(\mathbb{R}_0)\}$
- $\mathbb{G} := \{\mathcal{G}_t, t \geq 0\}$ where \mathcal{G}_t represents the information available to the investor at time t . We assume that $\mathcal{H}_t^\tau \subseteq \mathcal{G}_t$, i.e. the investor is instantaneously aware of default when it happens.

In this study, we consider that, once default occurs, the stock is not traded any more, which justifies the assumption above. Different would be the case in which by default is meant that the liabilities amount to a larger volume than the value of the firm itself. In this case, an instantaneous knowledge of default would most likely not be realistic. Note also that the default time τ is a totally inaccessible stopping time for the investor.

The investor's optimization problem is then to divide his money between the asset S_1 and the bond S_0 within a defined time horizon $T > 0$. The process $\pi(t)$, $t \in [0, T]$, represents the fraction of wealth invested in S_1 . Note that π is a \mathbb{G} -adapted stochastic process. In the following arguments we will discuss various relationships among the filtrations \mathbb{G} , \mathbb{F} , \mathbb{H}^Λ and \mathbb{H}^τ . Hence, in order to have one unique framework, we choose to model the investor's wealth $\tilde{X}_\pi(t)$, $t \in [0, T]$, as:

$$(1.6) \quad d\tilde{X}(t) = (1 - \pi(t))dS_0(t) + \pi(t)d^-S_1(t)$$

with initial value $\tilde{X}(0) = x_0 > 0$. Here $d^-S_1(t)$ represents a forward integral. We refer to [25, 26] for the treatment of the forward integral with respect to the Wiener process, and to [12] for the case of integration with respect to the compensated Poisson random measure. The forward integral is an extension of the Itô integral, but does not require the adaptedness of the integrands to the integral filtration. Applications of this type of integration to optimization problems and the justification of the use of this integrals from the modeling point of view have been studied. See e.g. [2, 13, 10, 20]. We also refer to [14] for a unified presentation of the topics.

The use of forward integral allows for no specification on the coefficients when it comes to adaptedness. We also remark that no a priori assumption of conditions typical of the framework of enlargement of filtrations are necessary. Hence, we can directly interpret the stock price dynamics as:

$$(1.7) \quad d^-S_1(t) = S_1(t-)\mathbf{1}_{\{\tau > t\}} \left(\mu(t)dt + \sigma(t)dW^-(t) + \int_{\mathbb{R}_0} \theta(t, z)d\tilde{N}(d^-t, dz) + \kappa(t)dH(t) \right), \quad S_1(0) > 0,$$

and we only assume that the coefficients μ, σ , and κ are càglàd stochastic processes and θ is a càglàd random field in the sense that $\theta(\cdot, z)$ is càglàd ν -a.e. (\mathbb{P} -a.e.). We also set:

$$\mathbb{E} \left[\int_0^T |\rho(s)| + |\mu(s)| + |\sigma(s)|^2 + \int_{\mathbb{R}_0} |\theta(s, z)|^2 \nu(dz) ds \right] < \infty.$$

Of course, in case of adapted coefficients to the corresponding filtrations, then the integrals here above would correspond to an Itô type of integration.

Since we want S_1 to stay positive before default and non-negative at all times, we assume

$$(1.8) \quad -1 < \theta(t, z, \omega) \quad (dt \times \nu(dz) \times d\mathbb{P} \text{ a.e.})$$

$$(1.9) \quad -1 \leq \kappa(t, \omega) < K \quad (dt \times d\mathbb{P} \text{ a.e.})$$

for some $-1 < K < \infty$. Using an adequate version of the Itô formula (Theorem 2.5), we see that the solution of (1.7) is

$$(1.10) \quad S_1(t) = S_1(0) \left(1 + \int_0^{t \wedge \tau} \kappa(s)dH(s) \right) \exp \left\{ \int_0^{t \wedge \tau} \left[\mu(s) - \frac{1}{2}\sigma^2(s) \right] ds \right\}$$

$$\begin{aligned}
& + \int_0^{t \wedge \tau} \sigma(s) d^-W(s) - \int_0^{t \wedge \tau} \int_{\mathbb{R}_0} [\ln(1 + \theta(s, z)) - \theta(s, z)] \nu(dz) ds \\
& + \int_0^{t \wedge \tau} \int_{\mathbb{R}_0} \ln(1 + \theta(s, z)) \tilde{N}(d^-s, dz) \Big\}
\end{aligned}$$

and it is easy to argue that this solution is unique. By application of the Itô formula again, we can see that the (unique) solution of (1.6), for a given admissible π (see Definition 3.1), is:

$$\begin{aligned}
\tilde{X}_\pi(t) = x_0 \exp \Big\{ & \int_0^{t \wedge \tau} \left[\rho(s) + (\mu(s) - \rho(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \right] ds \\
& + \int_0^{t \wedge \tau} \int_{\mathbb{R}_0} [\ln(1 + \pi(s)\theta(s, z)) - \pi(s)\theta(s, z)\nu(dz)] ds + \int_0^{t \wedge \tau} \sigma(s)\pi(s) d^-W(s) \\
& + \int_0^{t \wedge \tau} \int_{\mathbb{R}_0} \ln(1 + \pi(s)\theta(s, z)) \tilde{N}(d^-s, dz) + \int_0^{t \wedge \tau} \ln(1 + \kappa(s)\pi(s)) dH(s) \Big\}.
\end{aligned}$$

In this framework we study the optimal portfolio problem

$$(1.11) \quad \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[U(X_\pi(T))],$$

of an investor having \mathbb{G} as flow of information at disposal and U as utility function. Here $\mathcal{A}_{\mathbb{G}}$ represents the set of admissible portfolios (see Definition 3.1) and $X_\pi(t)$, $t \in [0, T]$, is an appropriately discounted wealth process. To explain, the focus of the paper is to optimize the portfolio up to the time of default or the time horizon T , whichever comes first. Should default occur before T , we will need to discount in order to compare the value of money at $\tau < T$ and at T . An arbitrary discount factor $d(\tau, T)$ can be used as long as $d(\tau, T) = 1$ for $\tau \geq T$. We use the interest earned in the risk free account as discount factor, i.e.

$$d(\tau, T) = \exp \left\{ \mathbf{1}_{\{\tau < T\}} \int_\tau^T \rho(s) ds \right\}.$$

For convenience we set $X_\pi(T) = d(\tau, T)\tilde{X}_\pi(T)$. Hence we have

$$(1.12) \quad X_\pi(T) = x_0 \exp \left\{ \mathbb{K}_{\{\tau < T\}} \int_\tau^T \rho(s) ds + \int_0^{T \wedge \tau} \left[\rho(s) + (\mu(s) - \rho(s))\pi(s) \right. \right.$$

$$\begin{aligned}
& -\frac{1}{2}\sigma^2(s)\pi^2(s)]ds + \int_0^{T\wedge\tau} \int_{\mathbb{R}_0} \left[\ln(1 + \pi(s)\theta(s, z)) - \pi(s)\theta(s, z)\nu(dz) \right] ds \\
& + \int_0^{T\wedge\tau} \int_{\mathbb{R}_0} \ln(1 + \pi(s)\theta(s, z)) \tilde{N}(d^-s, dz) \\
& + \int_0^{T\wedge\tau} \sigma(s)\pi(s)d^-W(s) + \int_0^{T\wedge\tau} \ln(1 + \kappa(s)\pi(s))dH(s) \Big\}.
\end{aligned}$$

Related to our optimization problem is the optimization of investments under uncertain time-horizons, as done in [7, 11]. In [7], optimization ends at a stopping time τ related to the noise in stock price. In [11] both optimal consumption and investment are treated. Typically the problems are solved using some variants of Hamilton-Jacobi-Bellman (HJB) equations. Our approach differs from these works for several reasons. First we focus on different streams of information for the investor, second we consider that the loss in case of default depends on the position in the risky asset. Moreover, our approach is different in framework and we do not use HJB type solutions. In [23] we find a study of a problem similar to ours. The approach is however entirely different as in this case backward stochastic differential equations are involved. Moreover we allow for a more general information structure and we consider a Lévy type of noise in the price dynamics.

Our work has some similarities to [1], where an optimization problem is considered when the stock dynamics include a jump component with an unknown intensity modeled by a continuous time Markov chain. But the filtering techniques therein may be less suited to default modeling since default is a jump happening only once. The methodology presented there relies on HJB equations and differs from ours.

Bielecki and coauthors consider various forms of optimal investments in e.g. [4], [6] and [3], looking at optimality and hedging when there is a number of instruments some of which are subject to default. However, their main focus is on the use of defaultable instruments for hedging purposes and the evaluation on whether to invest in defaultable bonds. In the same line is the study in [18].

As announced, in this paper we adopt the framework of anticipating stochastic calculus, specifically forward integration to tackle the optimization problem (1.11). Moreover, we consider the problem for various choices of investor's information flow \mathbb{G} . To the best of our knowledge it is the first time that the framework of forward integration is applied in optimization problems in presence of default.

In this paper we provide a characterization for the existence of locally optimal controls in a great generality both in the choice of utility function and in the amount of information available. Considerations on the meaning of locality are also provided. These topics are presented in Section 3. To achieve these results an expansion of the literature on forward integrals was needed: existence and convergence results and an extended version of the Itô formula adequate for our applications are presented in Section 2. This section is also of

mathematical interest independently of the application here treated. Some examples are given in Section 4.

2. MATHEMATICAL FRAMEWORK: FORWARD INTEGRALS

Forward integrals were introduced by Russo and Valois in the articles [25] and [26] for continuous processes and in [12] for pure jump Lévy process, see also [14] for a systematic presentation.

The forward integral is a type of stochastic anticipating integration that does not require assumptions of adaptedness or predictability to some filtration related to the integrator. Moreover, it is also an extension of the Itô integral in the sense that when the appropriate predictability is in place the two integrals coincide. This makes the forward integral especially suited for studying portfolio optimization problems under insider or partial information, where different filtrations are considered. See for e.g. [2, 12] and [14].

We follow the idea of [20] and consider the forward integral with respect to the Wiener processes as a limit in $L^1(\mathbb{P})$. This would also imply forward integrability in the sense of Russo and Valois, [25, 26, 27], who consider the same limit in probability.

Definition 2.1. *We say that the stochastic process $\sigma = \sigma(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, is forward integrable over the interval $[0, T]$ with respect to W if there exists a process $I = I(\sigma, t)$, $t \in [0, T]$, such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \sigma(s) \frac{W(s + \epsilon) - W(s)}{\epsilon} ds - I(\sigma, t) \right| \right] \longrightarrow 0, \quad \text{as } \epsilon \rightarrow 0^+,$$

In this case we write

$$I(\sigma, t) = \int_0^t \sigma(s) d^-W(s), \quad t \in [0, T],$$

and call $I(\sigma, t)$ the forward integral of σ with respect to W on $[0, t]$.

Lemma 2.2 gives us a connection between the Itô integral and the forward integral, where we see that the forward integral coincides with the Itô integral when the latter exists.

Lemma 2.2. *Let $\mathbb{G} = \{\mathcal{G}_t, t \in [0, T]\}$ be a given filtration. Suppose that*

- (1) *W is a semimartingale with respect to the filtration \mathbb{G} ,*
- (2) *σ is \mathbb{G} -predictable and the Itô integral $\int_0^T \sigma(t) dW(t)$ exists (in $L^1(\mathbb{P})$),*

then σ is forward integrable and

$$\int_0^T \sigma(t) d^-W(t) = \int_0^T \sigma dW(t).$$

For proof we refer to e.g. [14, Lemma 8.9].

Definition 2.3. *The forward integral*

$$J(\theta, t) := \int_0^t \int_{\mathbb{R}_0} \theta(s, z) \tilde{N}(d^-s, dz), \quad t \in [0, T]$$

with respect to the Poisson random measure \tilde{N} of a càglàd random field $\theta(t, z, \omega)$, $t \in [0, T]$, $z \in \mathbb{R}_0$, $\omega \in \Omega$, is defined as

$$J(\theta, t) = \lim_{m \rightarrow \infty} \int_0^t \int_{\mathbb{R}_0} \theta(s, z) \mathbf{1}_{U_m} \tilde{N}(ds, dz)$$

if the limit exists in $L^2(\mathbb{P})$. Here, U_m , $m = 1, 2, \dots$, is an increasing sequence of compact sets $U_m \subset \mathbb{R}_0$ with $\nu(U_m) < \infty$ such that $\lim_{m \rightarrow \infty} U_m = \mathbb{R}_0$.

The similar extension to Itô integrals is also true in this case, we have from [14, Remark 15.2]:

Remark 2.4. *Let $\mathbb{G} = \{\mathcal{G}_t, t \in [0, T]\}$ be a filtration such that*

- (1) *The process $\eta(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz)$, $t \in [0, T]$, is a semimartingale with respect to \mathbb{G} .*
- (2) *The random field $\theta = \theta(t, z)$, $t \in [0, T]$, $z \in \mathbb{R}_0$, is \mathbb{G} -predictable.*
- (3) *The integral $\int_0^t \int_{\mathbb{R}_0} \theta(t, z) \tilde{N}(ds, dz)$ exists as a classical Itô integral.*

Then the forward integral with respect to \tilde{N} also exists and we have

$$\int_0^T \int_{\mathbb{R}_0} \theta(s, z) \tilde{N}(d^-t, dz) = \int_0^T \int_{\mathbb{R}_0} \theta(s, z) \tilde{N}(dt, dz).$$

2.1. The Itô formula for forward integrals. An Itô formula for forward type integrals when the integrator is continuous was developed in [26, 27]. An Itô formula for forward integrals with Poisson random measures is found in [12], both the results are also summarized in [14]. In this paper we need a more general version that include processes of finite variation to guarantee the existence of solutions of (1.6) and (1.7). The proof can be seen as a continuation of the one presented in [14, Theorem 8.12], thus only the additional part is treated in detail.

Theorem 2.5.

Let

$$d^-X(t) = x + \mu(t)dt + \sigma(t)d^-W(t) + \int_{\mathbb{R}_0} \theta(t, z) \tilde{N}(d^-t, dz) + d\zeta(t),$$

where

- μ is a stochastic process satisfying $\int_0^T |\mu(s)|ds < \infty$ \mathbb{P} -a.s.
- σ is forward integrable with respect to W .

- θ and $|\theta|$ are forward integrable with respect to \tilde{N} and θ satisfies

$$\int_0^T \int_{\mathbb{R}_0} |\theta(s, z)|^2 \nu(dz) ds < \infty \quad \mathbb{P}\text{-a.s.}$$

- ζ is a càdlàg pure jump process of finite variation, with

$$(2.1) \quad \mathbb{P}\left(\text{There exist } t \in [0, T] \text{ such that } \Delta\zeta(t) > 0 \text{ and } N(\Delta t, U) > 0\right) = 0$$

for all $U \subset \mathbb{R}_0$ compact. Here $N(\Delta t, U) := N((0, t], U) - N((0, t), U)$ and $\Delta\zeta(t) := \zeta(t) - \zeta(t-)$.

Assume $f \in C^2(\mathbb{R})$ and let $Y(t) = f(X(t))$. Then

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t \left[f'(X(s-))\mu(s) + f''(X(s-))\sigma^2(s) \right] ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f(X(s-) + \theta(s, z)) - f(X(s-)) - f'(X(s-))\theta(s, z) \right] \nu(dz) ds \\ &+ \int_0^t f'(X(s-))\sigma(s)d^-W(s) + \int_0^t \int_{\mathbb{R}_0} \left[f(X(s-) + \theta(s, z)) - f(X(s-)) \right] \tilde{N}(ds, dz) \\ &+ \sum_{\substack{0 < s < t \\ \Delta\zeta(s) \neq 0}} \left[f(X(s-) + \Delta\zeta(s)) - f(X(s-)) \right]. \end{aligned}$$

Remark 2.6. Condition (2.1) is for instance fulfilled if N and ζ are independent.

Proof. Let

$$X_m(t) = x + \int_0^t \mu(s)ds + \int_0^t \sigma(s)d^-W(s) + \int_0^t \int_{\mathbb{R}_0} \mathbf{1}_{U_m}(z)\theta(s, z)\tilde{N}(ds, dz) + \zeta(t),$$

where $\mathbf{1}_{U_m}$ is as in Definition 2.3. We denote α_i , $i = 1, 2, \dots$ the times of the jumps of X_m . By condition (2.1) we can uniquely (\mathbb{P} -a.s.) divide the sequence α_i by the jumps of either ζ or $\mathbf{1}_{U_m}(z)N(dt, dz)$ as α_i^ζ and α_i^N . We formally set $\alpha_0 = \alpha_0^\zeta = \alpha_0^N = 0$.

Then

$$\begin{aligned} f(X_m(t)) - f(X_m(0)) &= \sum_i \left[f(X_m(\alpha_i \wedge t)) - f(X_m(\alpha_i \wedge t-)) \right] \\ &\quad + \sum_i \left[f(X_m(\alpha_i \wedge t-)) - f(X_m(\alpha_{i-1} \wedge t)) \right] \\ &= \sum_i \left[f(X_m(\alpha_i^\zeta \wedge t)) - f(X_m(\alpha_i^\zeta \wedge t-)) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_i \left[f(X_m(\alpha_i^N \wedge t)) - f(X_m(\alpha_i^N \wedge t-)) \right] \\
& + \sum_i \left[f(X_m(\alpha_i \wedge t-)) - f(X_m(\alpha_{i-1} \wedge t)) \right] \\
& = J_1(t) + J_2(t) + J_3(t),
\end{aligned}$$

with

$$J_1(t) = \sum_{\substack{0 < s \leq t \\ \Delta\zeta(t) \neq 0}} \left[f(X_m(s-) + \Delta\zeta(s)) - f(X_m(s-)) \right]$$

and

$$\begin{aligned}
(2.2) \quad J_2(t) &= \sum_i \left[f(X_m(\alpha_i^N)) - f(X_m(\alpha_i^N -)) \right] \mathbf{1}_{\{\alpha_i^N \leq t\}} \\
&= \int_0^t \int_{\mathbb{R}_0} \left[f(X_m(s-) + \theta(s, z)) - f(X_m(s-)) \right] N(ds, dz) \\
&= \int_0^t \int_{\mathbb{R}_0} \left[f(X_m(s-) + \theta(s, z)) - f(X_m(s-)) \right] \tilde{N}(ds, dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} \left[f(X_m(s-) + \theta(s, z)) - f(X_m(s-)) \right] \nu(dz) ds.
\end{aligned}$$

For the elements of the sum in $J_3(t)$ we use [14, Theorem 8.12]:

$$\begin{aligned}
J_3(t) &= \sum_i \left[\int_{\alpha_{i-1} \wedge t}^{\alpha_i \wedge t} \left[f'(X_m(s-)) \mu(s) ds - \int_{\mathbb{R}_0} f'(X_m(s-)) \mathbf{1}_{U_m} \theta(t, z) \nu(dz) \right] ds \right. \\
&\quad \left. + \int_{\alpha_{i-1} \wedge t}^{\alpha_i \wedge t} f'(X_m(s-)) \sigma(s) d^-W(s) + \int_{\alpha_{i-1} \wedge t}^{\alpha_i \wedge t} f''(X_m(s-)) \sigma^2(s) ds \right] \\
&= \int_0^t \left[f'(X_m(s-)) \mu(s) + f''(X_m(s-)) \sigma^2(s) - \int_{\mathbb{R}_0} f'(X(s-)) \mathbf{1}_{U_m} \theta(t, z) \nu(dz) \right] ds \\
&\quad + \int_0^t f'(X_m(s-)) \sigma(s) d^-W(s).
\end{aligned}$$

Adding J_1 , J_2 and J_3 together and letting $m \rightarrow \infty$ the result follows. □

2.2. Convergence results for forward integrals. We will need the following convergence for forward integrals for the proof of the forthcoming Theorem 3.3.

Lemma 2.7. *Suppose the stochastic process $\sigma = \sigma(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, is elementary, meaning that it has the form*

$$\sigma(t, \omega) = \sum_{i=0}^{N-1} \sigma_{t_i}(\omega) \mathbf{1}_{(t_i, t_{i+1}]}(t),$$

where $\sigma_{t_i} \in L^2(\mathbb{P})$ and $0 = t_0 < t_1 \cdots < t_N = T$. Then σ is forward integrable and

$$(2.3) \quad \int_0^t \sigma(s) d^-W(s) = \sum_{i=0}^{N-1} \sigma_{t_i} \mathbf{1}_{\{t_i < t\}} (W(t_{i+1} \wedge t) - W(t_i)), \quad t \in [0, T].$$

Proof. For simplicity in notation we only prove that

$$\lim_{\epsilon \rightarrow 0^+} \mathbb{E} \left[\sup_{0 \leq M \leq N} \left| \int_0^{t_M} \sigma(s) \frac{W(s+\epsilon) - W(s)}{\epsilon} ds - \sum_{i=0}^{M-1} \sigma_{t_i} (W(t_{i+1}) - W(t_i)) \right| \right] = 0.$$

As we will see during the calculations, the general case (which would consider $\sup_{t \in [0, T]}$) is identical but has more cumbersome notation. Denote

$$\begin{aligned} K_\epsilon &= \mathbb{E} \left[\sup_{0 \leq M \leq N} \left| \int_0^{t_M} \sigma(s) \frac{W(s+\epsilon) - W(s)}{\epsilon} ds - \sum_{i=0}^{M-1} \sigma_{t_i} (W(t_{i+1}) - W(t_i)) \right| \right] \\ &= \mathbb{E} \left[\sup_{0 \leq M \leq N} \left| \sum_{i=0}^{M-1} \sigma_{t_i} \left(\int_{t_i}^{t_{i+1}} \frac{1}{\epsilon} \int_s^{s+\epsilon} dW(u) ds - \int_{t_i}^{t_{i+1}} dW(u) \right) \right| \right] \\ &= \mathbb{E} \left[\sup_{0 \leq M \leq N} \left| \sum_{i=0}^{M-1} \sigma_{t_i} \left(\frac{1}{\epsilon} \int_{t_i}^{t_{i+1}} \int_{u-\epsilon}^u ds dW(u) + \frac{1}{\epsilon} \int_{t_{i+1}-\epsilon}^{t_{i+1}} \int_{t_i}^{s+\epsilon} dW(u) ds \right. \right. \right. \\ &\quad \left. \left. - \frac{1}{\epsilon} \int_{t_i-\epsilon}^{t_i} \int_{t_i}^{s+\epsilon} dW(u) ds - \int_{t_i}^{t_{i+1}} dW(s) \right) \right| \right] \\ &= \mathbb{E} \left[\sup_{0 \leq M \leq N} \left| \sum_{i=0}^{M-1} \frac{\sigma_{t_i}}{\epsilon} \left(\int_{t_{i+1}-\epsilon}^{t_{i+1}} \int_{t_i}^{s+\epsilon} dW(u) ds - \int_{t_i-\epsilon}^{t_i} \int_{t_i}^{s+\epsilon} dW(u) ds \right) \right| \right] \\ (2.4) \quad &\leq \mathbb{E} \left[\sup_{0 \leq M \leq N} \left\{ \sum_{i=0}^{M-1} \frac{|\sigma_{t_i}|}{\epsilon} \left(\int_{t_{i+1}}^{t_{i+1}+\epsilon} |W(s) - W(t_{i+1})| ds + \int_{t_i}^{t_i+\epsilon} |W(s) - W(t_i)| ds \right) \right\} \right] \end{aligned}$$

It is clear that the supremum in (2.4) is attained for $M = N$. Hence

$$\begin{aligned}
K_\epsilon &\leq \sum_{i=0}^{N-1} \mathbb{E} \left[\frac{1}{\epsilon} \int_{t_{i+1}}^{t_{i+1}+\epsilon} |\sigma_{t_i}| |W(s) - W(t_{i+1})| ds \right] + \sum_{i=0}^{N-1} \mathbb{E} \left[\frac{1}{\epsilon} \int_{t_i}^{t_i+\epsilon} |\sigma_{t_i}| |W(s) - W(t_i)| ds \right] \\
&\leq \sum_{i=0}^{N-1} \sqrt{\mathbb{E} \left[\int_{t_{i+1}}^{t_{i+1}+\epsilon} \sigma_{t_i}^2 ds \right]} \sqrt{\mathbb{E} \left[\int_{t_{i+1}}^{t_{i+1}+\epsilon} \left(\frac{W(s) - W(t_{i+1})}{\epsilon} \right)^2 ds \right]} \\
&\quad + \sum_{i=0}^{N-1} \sqrt{\mathbb{E} \left[\int_{t_i}^{t_i+\epsilon} \sigma_{t_i}^2 ds \right]} \sqrt{\mathbb{E} \left[\int_{t_i}^{t_i+\epsilon} \left(\frac{W(s) - W(t_i)}{\epsilon} \right)^2 ds \right]} \\
&= \sqrt{\epsilon} \frac{2}{\sqrt{2}} \sum_{i=0}^{N-1} \sqrt{\mathbb{E} [\sigma_{t_i}^2]}.
\end{aligned}$$

Which vanish when $\epsilon \rightarrow 0^+$. Note that the right hand side of (2.3) is an element of $L^1(\mathbb{P})$ from Hölder's inequality. \square

The result in the forthcoming Lemma 2.8 was inspired by [2], where a similar result was proved with convergence in probability. There are also results similar to Lemmas 2.7 and 2.8 in [20], but with different assumptions that depend on Malliavin derivatives.

Theorem 2.8. *Assume that the stochastic process $\sigma = \sigma(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, is bounded, càglàd and forward integrable. Then there exists a sequence of elementary functions σ_n , $n = 1, 2, \dots$ such that*

$$\int_0^T \sigma_n(t) d^-W(t) \longrightarrow \int_0^T \sigma(t) d^-W(t), \quad \text{as } n \rightarrow \infty.$$

Proof. Since σ is càglàd it can be approximated by elementary functions uniformly in t and pointwise in ω . Let σ_n , $n = 1, 2, \dots$, be a sequence of such elementary functions. Note that by Lemma 2.7, all the σ_n are forward integrable.

▷ Define Y as the Banach spanned by $\sigma, \sigma_1, \sigma_2, \dots$, with norm

$$\|f\|_Y = \sqrt{\mathbb{E} \left[\left(\sup_{t \in [0, T]} |f(t)| \right)^2 \right]},$$

▷ Define a family of operators by $I_\epsilon : Y \rightarrow L^1(\mathbb{P})$, $\epsilon \in (0, 1)$ by

$$I_\epsilon(f) = \int_0^T f(s) \frac{W(s+\epsilon) - W(s)}{\epsilon} ds.$$

Note that

$$\begin{aligned}
\|I_\epsilon(f)\|_{L^1(\mathbb{P})} &= \mathbb{E} \left[\left| \int_0^T f(s) \frac{W(s+\epsilon) - W(s)}{\epsilon} ds \right| \right] \\
&\leq \sqrt{\mathbb{E} \left[\int_0^T (f(s))^2 ds \right]} \sqrt{\mathbb{E} \left[\int_0^T \left(\frac{W(s+\epsilon) - W(s)}{\epsilon} \right)^2 ds \right]} \\
(2.5) \quad &= \sqrt{\frac{T}{\epsilon}} \sqrt{\mathbb{E} \left[\int_0^T (f(s))^2 ds \right]} \leq \frac{T}{\sqrt{\epsilon}} \|f\|_Y.
\end{aligned}$$

So for ϵ fixed, I_ϵ is a bounded linear operator from Y to $L^1(\mathbb{P})$.

▷ Let $\delta > 0$ be fixed. Since $f \in Y$ are forward integrable, there exists $\epsilon_0(\delta)$ such that

$$(2.6) \quad \|I_\epsilon(f)\|_{L^1(\mathbb{P})} \leq \left\| \int_0^T f(s) d^-W(s) \right\|_{L^1(\mathbb{P})} + \delta$$

for all $0 < \epsilon < \epsilon_0(\delta)$. Combining (2.5) and (2.6), we can conclude that the family $\{\|I_\epsilon(f)\|_{L^1(\mathbb{P})} \mid \epsilon \in (0, 1)\}$ is uniformly bounded for all $f \in Y$. Thus, by the Banach-Steinhaus theorem [28, Theorem 4.52], there exists a $K < \infty$ such that $\{\|I_\epsilon\| \mid \epsilon \in (0, 1)\} < K$.

▷ By the previous arguments we can conclude that:

$$\begin{aligned}
\left\| \int_0^T \sigma(t) d^-W - \int_0^T \sigma_n(t) d^-W(t) \right\|_{L^1(\mathbb{P})} &= \lim_{\epsilon \rightarrow 0^+} \|I_\epsilon(\sigma - \sigma_n)\|_{L^1(\mathbb{P})} \\
&\leq K \|\sigma - \sigma_n\|_Y \longrightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

□

Remark 2.9. Combining Lemmas 2.7 and 2.8, we see that if σ is bounded, càglàd and forward integrable, then for any $t \in [0, T]$

$$\int_0^t \sigma(s) d^-W(s) = \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbf{1}_{\{t_i < t\}} \sigma(t_i) (W(t_{i+1} \wedge t) - W(t_i))$$

where $0 = t_0 < t_1 < \dots < t_N = T$ are partitions of $[0, T]$, and the limit is taken over partitions such that $\Delta t := \sup_i (t_{i+1} - t_i) \rightarrow 0$, with convergence in $L^1(\mathbb{P})$.

We have the following convergence result for forward integrals with respect to Poisson random measures:

Theorem 2.10. *Let $\beta_i(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, $i = 1, 2, \dots$ be a sequence of bounded stochastic processes. Assume $\beta_i \rightarrow \beta$ for $i \rightarrow \infty$, pointwise in ω and uniformly in t , and β bounded.*

Let θ , $\beta\theta$ and, for all i , $\beta_i\theta$ be forward integrable with respect to \tilde{N} . Then

$$\int_0^T \int_{\mathbb{R}_0} \beta_i(s) \theta(s, z) \tilde{N}(d^-s, dz) \rightarrow \int_0^T \int_{\mathbb{R}_0} \beta(s) \theta(s, z) \tilde{N}(d^-s, dz)$$

in $L^1(\mathbb{P})$ as $i \rightarrow \infty$.

Proof. Consider the Banach space Y spanned by $\{\beta, \beta_1, \beta_2, \dots\}$ equipped with norm

$$\|f\|_Y = \sqrt{\mathbb{E} \left[\left(\sup_{t \in [0, T]} |f(t)| \right)^2 \right]}.$$

We define the operator $J : Y \rightarrow L^1(\mathbb{P})$ as

$$J(f) = \int_0^T f(s) \theta(s, z) N(d^-s, dz), \quad f \in Y$$

and recall that $\|J(f)\|_{L^1(\mathbb{P})} < \infty$ since all $f \in Y$ are forward integrable. We also define the operators $J_m : Y \rightarrow L^1(\mathbb{P})$, $m = 1, 2, \dots$ by

$$J_m(f) = \int_0^T \int_{\mathbb{R}_0} f(s) \theta(s, z) \mathbf{1}_{U_m}(z) \tilde{N}(ds, dz),$$

with $\mathbf{1}_{U_m}$ as described in Definition 2.3.

For every m , J_m is a bounded linear operator. To prove boundedness, consider $X_m(t) = \int_0^t \int_{\mathbb{R}_0} \theta(s, z) \mathbf{1}_{U_m}(z) \tilde{N}(ds, dz)$. Note that $J_m(1) = X_m(T)$. Since X_m is a process of finite variation we can define a new process V_m as the total variation process $|X_m|$. A description of total variation processes can be found in [24, section 7, chapter I]. Then, with $f \in Y$,

$$\begin{aligned} \|J_m(f)\|_{L^1(\mathbb{P})} &= \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{R}_0} f(s) \theta(s, z) \mathbf{1}_{U_m}(z) \tilde{N}(ds, dz) \right| \right] \\ &\leq \mathbb{E} \left[\sup_{t \in [0, T]} |f(t)| V_m(T) \right] \\ &\leq \sqrt{\mathbb{E} \left[\left(\sup_{t \in [0, T]} |f(t)| \right)^2 \right]} \sqrt{\mathbb{E} \left[(V_m(T))^2 \right]} \\ &= \|f\|_Y \sqrt{\mathbb{E} \left[(V_m(T))^2 \right]} = \|f\|_Y A_m \end{aligned}$$

by Hölder's inequality. We can prove that A_m is finite by using the distributional properties of the Poisson random measure.

Since all the elements of Y are forward integrable, then the set $\{\|J_m(f)\|_{L^1(\mathbb{P})} : m = 1, 2, \dots\}$ is bounded for every $f \in Y$. By the Banach-Steinhaus Theorem, [28, Theorem 4.52], it follows that $\{\|J_m\| : m = 1, 2, \dots\}$ is bounded, i.e. there exists K such that

$$\sup_{m \in \mathbb{N}} \|J_m(f)\|_{L^1(\mathbb{P})} \leq K \|f\|_Y.$$

We are now ready to show the convergence result. Let $\epsilon > 0$ be given. Then

▷ Since $\beta\theta$ forward integrable there exists M_1 such that for $m > M_1$

$$\|J(\beta) - J_m(\beta)\|_{L^1(\mathbb{P})} < \frac{\epsilon}{3}.$$

▷ Since $\beta_i \rightarrow \beta$ in Y , there exists I such that for every $i > I$

$$\|\beta - \beta_i\|_Y < \frac{\epsilon}{3K},$$

thus also

$$\|J_m(\beta) - J_m(\beta_i)\|_{L^1(\mathbb{P})} < \frac{\epsilon}{3}$$

for all $m = 1, 2, \dots$.

▷ Since β_i is forward integrable, there exists M_2 such that for $m > M_2$

$$\|J_m(\beta_i) - J(\beta_i)\|_{L^1(\mathbb{P})} < \frac{\epsilon}{3}$$

The convergence result then follow by choosing $i > I$ and $m > \max(M_1, M_2)$. \square

3. OPTIMIZATION PROBLEM: LOCAL MAXIMUMS

Now we are ready to tackle directly our stated optimization problem (1.11). First we give a description of the set of the investor's admissible portfolios.

Definition 3.1. *The set $\mathcal{A}_{\mathbb{G}}$ of admissible portfolios consists of stochastic processes $\pi = \pi(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, such that*

i) π is càglàd and \mathbb{G} -adapted

ii) for every $\pi \in \mathcal{A}_{\mathbb{G}}$, there exists $\epsilon_\pi > 0$ such that for all t ,

$$(3.1) \quad \pi(t)\kappa(t) > -1 + \epsilon_\pi$$

and

$$(3.2) \quad \pi(t)\theta(t, z) > -1 + \epsilon_\pi$$

iii)

$$\mathbb{E} \left[\int_0^T |(\mu(s) - \rho(s))| |\pi(s)| + \sigma^2(s) \pi^2(s) ds \right] < \infty$$

and

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}_0} |\theta(s, z) \pi(s, z)|^2 \nu(dz) ds \right] < \infty$$

- iv) $\pi\sigma$ is càglàd and forward integrable with respect to W
- v) $\pi\theta$, $\ln(1 + \pi\theta)$ and $\frac{\pi\theta}{1+\pi\theta}$ are càglàd and forward integrable with respect to \tilde{N} .

In particular we note that condition i) ensures that the portfolio choices correspond to the investors knowledge and that condition ii) ensures that the investor never reaches zero wealth from the jumps of H or \tilde{N} , thus that our given solution is as stated in (1.12). In addition ii) means that fractions of the form $\frac{1}{1+\kappa\pi}$ are bounded, which is implicitly used in some forthcoming equations.

Note that if

$$\pi(s, \omega) = \alpha(\omega) \mathbf{1}_{(t, t+h]}(s),$$

where α is a bounded \mathcal{G}_t -measurable random variable, then $\pi \in \mathcal{A}_{\mathbb{G}}$ as long as (3.1) and (3.2) are satisfied.

As announced we are interested in the problem

$$(3.3) \quad \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[U(X_{\pi}(T))].$$

In general we consider utility functions that are increasing, differentiable and satisfy the forthcoming $A_{u.i.}$. We will search for solutions to (3.3) that are optimal in the sense that they cannot be improved by small perturbations.

Definition 3.2. We say that the stochastic process π is a local maximum for the problem (3.3) if

$$(3.4) \quad \mathbb{E}[U(X_{\pi+y\beta}(T))] \leq \mathbb{E}[U(X_{\pi}(T))]$$

for all bounded $\beta \in \mathcal{A}_{\mathbb{G}}$ and $|y| < \delta$ for some $\delta > 0$ that may depend on β .

From the terminology point of view, when we say that a *property holds under* (\mathbb{Q}, \mathbb{G}) , we mean that the property holds under the measure \mathbb{Q} with respect to the filtration \mathbb{G} . Moreover, we say that a stochastic process $Y(t)$ has the *martingale property* under (\mathbb{Q}, \mathbb{G}) if

$$\mathbb{E}_{\mathbb{Q}}[Y(t+h) - Y(t) | \mathcal{G}_t] = 0$$

for all $0 < t < t+h < \infty$.

Following the techniques in [2, 13], we consider perturbations of stochastic controls to find necessary and sometimes sufficient criteria to characterize local maximums. We will need the following assumption.

Assumption $A_{u.i.}$. We say that assumption $A_{u.i.}$ holds if for all $\pi \in \mathcal{A}_{\mathbb{G}}$

- i) $\mathbb{E}[U(X_{\pi}(T))] < \infty$
- ii) $0 < \mathbb{E}[U'(X_{\pi}(T))X_{\pi}(T)] < \infty$, with $U'(x) = \frac{dU}{dx}$

iii) For all $\pi, \beta \in \mathcal{A}_{\mathbb{G}}$ with β bounded, there exists $\delta > 0$ such that the family

$$(3.5) \quad \left\{ U'(X_{\pi+y\beta}(T)) X_{\pi+y\beta}(T) |\Psi(y, \beta, \pi)| \right\}_{y \in (-\delta, \delta)}$$

is uniformly integrable, where

$$(3.6) \quad \begin{aligned} \Psi(y, \beta, \pi) := & \int_0^{T \wedge \tau} \beta(s) [\mu(s) - \rho(s) - (\pi(s) + y\beta(s))\sigma^2(s)] ds \\ & + \int_0^{T \wedge \tau} \int_{\mathbb{R}_0} \left[\frac{\beta(s)\theta(s, z)}{1 + (\pi(s) + y\beta(s))\theta(s, z)} - \beta(s)\theta(s, z) \right] \nu(dz) ds \\ & + \int_0^{T \wedge \tau} \beta(s)\sigma(s) d^-W(s) + \int_0^{T \wedge \tau} \int_{\mathbb{R}_0} \frac{\beta(s)\theta(s, z)}{1 + (\pi(s) + y\beta(s))\theta(s, z)} \tilde{N}(d^-z, ds) \\ & + \int_0^{T \wedge \tau} \frac{\beta(s)\kappa(s)}{1 + \kappa(s)(\pi(s) + y\beta(s))} dH(s). \end{aligned}$$

Assumption $A_{u.i.}$ depends on the utility function U , in some ways the conditions are also a limitation on which utilities we can find solutions for. Item ii) is used when we do a change of measure at (3.13).

Condition iii) ensures we can use the desired technique to find local maximums. It is a necessary ingredient for Theorem 3.3. Uniform integrability is the minimal condition for taking limits under the integral sign in the framework adopted. Condition iii) is unfortunate in that it stems from mathematical rather than modeling necessities, but we cannot do without it.

There is a good discussion when a uniform integrability condition like the one in Assumption $A_{u.i.}$ is fulfilled in [14, section 16.5]. The conclusions from [14, section 16.5] can be transferred to our model. In fact, the presence of the dH integral does not influence these results.

Theorem 3.3. *Suppose $A_{u.i.}$ holds, $\pi \in \mathcal{A}_{\mathbb{G}}$ and the utility function U is increasing and differentiable.*

i) *If π is a local maximum for (3.3), then the process $M_\pi(t)$, $t \in [0, T]$, has the martingale property under $(\mathbb{Q}_\pi, \mathbb{G})$. Where M_π is defined as*

$$\begin{aligned} M_\pi(t) := & \int_0^{t \wedge \tau} \left[\mu(s) - \rho(s) - \pi(s)\sigma^2(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s, z)}{1 + \pi(s)\theta(s, z)} \nu(dz) \right] ds \\ & + \int_0^{t \wedge \tau} \sigma(s) d^-W(s) + \int_0^{t \wedge \tau} \int_{\mathbb{R}_0} \frac{\theta(s, z)}{1 + \pi(s)\theta(s, z)} \tilde{N}(d^-s, dz) \end{aligned}$$

$$(3.7) \quad + \int_0^{t \wedge \tau} \frac{\kappa(s)}{1 + \kappa(s)\pi(s)} dH(s),$$

and the measure \mathbb{Q}_π is defined by $d\mathbb{Q}_\pi = F_\pi(T)d\mathbb{P}$, with

$$(3.8) \quad F_\pi(T) = \frac{U'(X_\pi(T))X_\pi(T)}{\mathbb{E}[U'(X_\pi)X_\pi(T)]}.$$

ii) Suppose the mapping

$$(3.9) \quad y \rightarrow \mathbb{E}[U(X_{\pi+y\beta}(T))]$$

is concave for all bounded controls $\beta \in \mathcal{A}_\mathbb{G}$ and $|y| < \delta$, for some $\delta > 0$ that may depend on π . Then the converse is also true: π is a local maximum for (3.3) only if M_π has the martingale property under $(\mathbb{Q}_\pi, \mathbb{G})$.

Proof. Part i) If π is a local maximum, then for all bounded β we have

$$(3.10) \quad 0 = \frac{d}{dy} \mathbb{E}[U(X_{\pi+y\beta}(T))]_{|y=0} = \mathbb{E}[U'(X_{\pi+y\beta}(T)) \frac{d}{dy} X_{\pi+y\beta}(T)]_{|y=0}.$$

Here assumption $A_{u,i}$ is used, see for instance [15, Appendix A]. With some calculations we obtain

$$(3.11) \quad \begin{aligned} 0 &= \mathbb{E} \left[U'(X_\pi(T)) X_\pi(T) \left\{ \int_0^{T \wedge \tau} \beta(s) [\mu(s) - \rho(s) - \pi(s)\sigma^2(s)] ds \right. \right. \\ &\quad + \int_0^{T \wedge \tau} \int_{\mathbb{R}_0} \beta(s) \frac{-\pi(s)\theta^2(s, z)}{1 + \pi(s)\theta(s, z)} \nu(dz) ds + \int_0^{T \wedge \tau} \beta(s) \sigma(s) d^-W(s) \\ &\quad \left. \left. + \int_0^{T \wedge \tau} \int_{\mathbb{R}_0} \frac{\beta(s)\theta(s, z)}{1 + \pi(s)\theta(s, z)} \tilde{N}(d^-z, ds) + \int_0^{T \wedge \tau} \frac{\beta(s)\kappa(s)}{1 + \pi(s)\kappa(s)} dH(s) \right\} \right] \\ &= \mathbb{E} \left[U'(X_\pi(T)) X_\pi(T) \Psi(0, \beta, \pi) \right]. \end{aligned}$$

We now let $\beta(s) = \alpha \mathbf{1}_{(t, t+h]}(s)$, where α is a \mathcal{G}_t -measurable bounded random variable. We can put α outside the forward integrals, see for instance [14, Lemma 8.7] and [14, Remark 15.3] to get

$$\mathbb{E} \left[U'(X_\pi(T)) X_\pi(T) \left\{ \int_t^{(t+h) \wedge \tau} [\mu(s) - \rho(s) - \pi(s)\sigma^2(s)] ds \right. \right.$$

$$\begin{aligned}
& - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s, z)}{1 + \pi(s)\theta(s, z)} \nu(dz) \Big] ds + \int_t^{(t+h) \wedge \tau} \sigma(s) d^-W(s) \\
(3.12) \quad & + \int_t^{(t+h) \wedge \tau} \int_{\mathbb{R}_0} \frac{\theta(s, z)}{1 + \pi(s)\theta(s, z)} \tilde{N}(d^-z, ds) + \int_t^{(t+h) \wedge \tau} \frac{\kappa(s)}{1 + \kappa(s)\pi(s)} dH(s) \Big\} \mathbf{1}_{\{\tau > t\}} \alpha \Big] = 0.
\end{aligned}$$

Since this holds for all α , we conclude that

$$\mathbb{E}[F_\pi(T)(M_\pi(t+h) - M_\pi(t)) | \mathcal{G}_t] = 0$$

with $F_\pi(T)$ and M_π defined as in (3.8) and (3.7) respectively. Since $\mathbb{E}[F_\pi(T)] = 1$, we can define a new probability measure by

$$(3.13) \quad d\mathbb{Q}_\pi = F_\pi(T) d\mathbb{P}.$$

We thus have that π is a local maximum if M_π has the martingale property under $(\mathbb{Q}_\pi, \mathbb{G})$.

Part ii). To get the reverse conclusion, suppose M_π has the martingale property under $(\mathbb{Q}_\pi, \mathbb{G})$. Then, for $0 < t < t+h < T$,

$$\mathbb{E}_{\mathbb{Q}_\pi}[M_\pi(t+h) - M_\pi(t) | \mathcal{G}_t] = 0,$$

which is equivalent to

$$0 = \mathbb{E}[F_\pi(T)(M_\pi(t+h) - M_\pi(t))\alpha] = \mathbb{E}[F_\pi(T)\Psi(0, \alpha \mathbf{1}_{(t, t+h]}, \pi)]$$

for all bounded \mathcal{G}_t -measurable α . Which is the same as (3.12). Taking linear combinations of random variables α_i for different t, h , we see that (3.11) holds for all elementary processes $\beta \in \mathcal{A}_\mathbb{G}$, i.e. $\mathbb{E}[F_\pi(T)\Psi(0, \beta, \pi)] = 0$.

Let $\beta \in \mathcal{A}_\mathbb{G}$, β be bounded, and $\beta_j \in \mathcal{A}_\mathbb{G}$ be a sequence of elementary stochastic processes $\beta_j \in \mathcal{A}_\mathbb{G}$ such β_j converges pointwise in ω and uniformly in t to β . Then consider Ψ as in (3.6). We have

- ▷ $\Psi(0, \beta_j, \pi) \rightarrow \Psi(0, \beta, \pi)$, as $j \rightarrow \infty$, in $L^1(\mathbb{P})$. In fact, for d^-W integral we can apply Theorem 2.8, since $\beta\sigma$ is bounded and càglàd by assumption. For the $N(d^-t, dz)$ integral we apply Theorem 2.10. It is clear for the integrals ds and dH .

- ▷ Let $F_\pi^n(T) := \min(F_\pi(T), n)$. Since $F_\pi^n(T)$ is bounded, for every n , we have

$$F_\pi^n(T)\Psi(0, \beta_j, \pi) \rightarrow F_\pi^n(T)\Psi(0, \beta, \pi) \quad \text{in } L^1(\mathbb{P}) \text{ as } j \rightarrow \infty.$$

- ▷ The function $F_\pi(T)\Psi(0, \beta, \pi)$ is integrable by Assumption $A_{u.i.}$. Hence we can use dominated convergence to get

$$F_\pi^n(T)\Psi(0, \beta, \pi) \rightarrow F_\pi(T)\Psi(0, \beta, \pi) \quad \text{in } L^1(\mathbb{P}) \text{ as } n \rightarrow \infty.$$

Hence (3.11) holds for all $\beta \in \mathcal{A}_\mathbb{G}$. Since the mapping $y \rightarrow \mathbb{E}[U(X_{\pi+y\beta}(T))]$ is concave, (3.11) can only be zero if π is a local maximum. \square

Remark 3.4. If the process $M_\pi(t)$, $t \in [0, T]$, in Theorem 3.3 is adapted to \mathbb{G} , it is $(\mathbb{Q}_\pi, \mathbb{G})$ -martingale.

With the introduction of the forthcoming assumption A_{d^2} we can detail additional results on the convexity of (3.9) and the uniqueness of local maximums.

Assumption A_{d^2} : The utility function U is twice differentiable, strictly increasing and concave. Furthermore, for all $\pi, \beta \in \mathcal{A}_{\mathbb{G}}$ with β bounded, there exists a $\delta > 0$ such that the family

$$\left\{ U''(X_{\pi+y\beta}(T))X_{\pi+y\beta}^2(T)\Psi^2(y, \beta, \pi) + U'(X_{\pi+y\beta}(T))X_{\pi+y\beta}(T)[\Psi(y, \beta, \pi) + \Psi_y(y, \beta, \pi)] \right\}_{y \in (-\delta, \delta)}$$

is uniformly integrable where $\Psi(y, \beta, \pi)$ is defined in (3.6) and

$$\begin{aligned} \Psi_y(y, \beta, \pi) &:= \frac{d}{dy}\Psi(y, \beta, \pi) \\ &= - \int_0^{T \wedge \tau} \beta^2(s)\sigma^2(s)ds - \int_0^{T \wedge \tau} \int_{\mathbb{R}_0} \frac{\beta^2(s)\theta^2(s, z)}{[1 + (\pi(s) + y\beta(s))\theta(s, z)]^2} N(d^-s, dz) \\ &\quad - \int_0^{T \wedge \tau} \frac{\beta^2(s)\kappa^2(s)}{[1 + (\pi(s) + y\beta(s))\kappa(s)]^2} dH(s). \end{aligned} \quad (3.14)$$

Since it is reasonable to assume that the coefficients σ , θ and κ are not null on the same time intervals, then $\Psi_y(y, \beta, \pi) < 0$ for $y \in (-\delta, \delta)$ and $\beta \neq 0$.

Lemma 3.5 will give us a simple sufficient condition for the concavity of (3.9) in Theorem 3.3. Note that the equations in the proof may also be useful to prove concavity if the conditions in the lemma are not fulfilled.

Lemma 3.5. *Suppose A_{d^2} holds and the utility function U satisfies*

$$(3.15) \quad xU''(x) + U'(x) \leq 0 \quad \text{for all } x > 0.$$

Then the mapping (3.9), $y \rightarrow \mathbb{E}[U(X_{\pi+y\beta}(T))]$, $y \in (-\delta, \delta)$, $\delta > 0$, is concave for all $\pi \in \mathcal{A}_{\mathbb{G}}$ and bounded controls $\beta \in \mathcal{A}_{\mathbb{G}}$.

Proof. By assumptions $A_{u.i.}$ and A_{d^2} the following equations hold true:

$$\begin{aligned} \frac{d^2}{dy^2}\mathbb{E}[U(X_{\pi+y\beta}(T))] &= \\ &= \frac{d}{dy}\mathbb{E}\left[\left(U'(X_{\pi+y\beta}(T))X_{\pi+y\beta}(T)\Psi(y, \beta, \pi)\right)\right] \\ &= \mathbb{E}\left[X_{\pi+y\beta}(T)\Psi^2(y, \beta, \pi)\left(U''(X_{\pi+y\beta}(T))X_{\pi+y\beta}(T) + U'(X_{\pi+y\beta}(T))\right)\right. \\ &\quad \left.+ U'(X_{\pi+y\beta}(T))X_{\pi+y\beta}(T)\Psi_y(y, \beta, \pi)\right]. \end{aligned} \quad (3.16)$$

Thanks to (3.15) and the observation that $\Psi_y(y, \beta, \pi) < 0$ for all $y \in (-\delta, \delta)$, both summands are negative and the mapping (3.9) is locally concave. \square

Remark 3.6. *Examples of utility functions satisfying (3.15) are the power utility $U(x) = \frac{1}{1-c}x^{1-c}$ when $c > 1$, and logarithmic utility $U(x) = \log(x)$, while the exponential utility, $U(x) = \frac{-1}{\gamma}e^{-\gamma x}$, does not.*

Remark 3.7. *Condition (3.15) can also be discussed in terms of the Arrow Pratt measure of relative risk aversion. This measure is defined by*

$$R_u(x) = \frac{-xU''(x)}{U'(x)},$$

so an equivalent way of stating condition (3.15) would be to require the $R_u(x) \geq 1$.

We can use a concavity argument from the derivatives to get some form of uniqueness. A similar argument occurs in [20], where it is proven that local maximums are unique in the case of logarithmic utility under some restriction on admissible controls. In our case we have the following result:

Theorem 3.8. *Suppose A is a convex set in $\mathcal{A}_{\mathbb{G}}$ such that all $\pi \in A$ are bounded. If A_{d^2} , $A_{u.i.}$ and (3.15) hold, then there can at most be one local maximum in A .*

Proof. Suppose $\pi_1, \pi_2 \in A$ are two local maximums. Let $\pi_2 - \pi_1 = \beta$. Since A is convex, we have $\pi_1 + y\beta \in A$ for $y \in [0, 1]$. We note that

$$\frac{d}{dy}\mathbb{E}[U(X_{\pi_1+y\beta}(T))]\big|_{y=a} = \frac{d}{dy}\mathbb{E}[U(X_{(\pi_1+a\beta)+\zeta\beta}(T))]\big|_{\zeta=0} \quad \text{for } a \in [0, 1].$$

By assumption $A_{u.i.}$ and A_{d^2} we can give an evaluation of the first (here above) and also the second derivative.

We show that there cannot exist two local maximums by contradiction. Consider

(3.17)

$$\frac{d}{dy}\mathbb{E}[U(X_{\pi_1+y\beta}(T))]\big|_{y=1} = \frac{d}{dy}\mathbb{E}[U(X_{\pi_1+\beta+\zeta\beta}(T))]\big|_{\zeta=0} = \frac{d}{dy}\mathbb{E}[U(X_{\pi_2+\zeta\beta}(T))]\big|_{\zeta=0} = 0$$

since $\pi_1 + \beta = \pi_2$, and π_2 is a local maximum. On the other hand, we also have that π is a local maximum, hence

$$\frac{d}{dy}\mathbb{E}[U(X_{\pi_1+y\beta}(T))]\big|_{y=0} = 0.$$

Since $\frac{d^2}{dy^2}\mathbb{E}[U(X_{\pi_1+y\beta}(T))]\big|_{y=0} < 0$ as shown in Lemma 3.5, then $\frac{d}{dy}\mathbb{E}[U(X_{\pi_1+y\beta}(T))]$ is monotone and it can only be zero at one point. \square

In case some adaptedness is present in the model, then the results of Theorem 3.3 take a different interesting form.

Theorem 3.9. *Suppose that μ, σ, θ and λ are \mathbb{G} -adapted processes and random fields, assumption $A_{u.i.}$ holds and $\mathcal{F}_t \vee \mathcal{H}_t^\lambda \subseteq \mathcal{G}_t$ for all $t \in [0, T]$.*

i) *If π is a local maximum, then $M_\pi(t)$, $t \in [0, T]$, is a martingale under $(\mathbb{Q}_\pi, \mathbb{G})$.*

ii) If π is a local maximum, then the stochastic process

$$\hat{M}_\pi(t) = M_\pi(t) - \int_0^t \frac{1}{Z(s)} d[M_\pi, Z](s), \quad t \in [0, T],$$

is a martingale under (\mathbb{P}, \mathbb{G}) , where

$$Z(t) = \mathbb{E} \left[\frac{d\mathbb{P}}{d\mathbb{Q}_\pi} \middle| \mathcal{G}_t \right] = \left(\mathbb{E}[F_\pi(T) | \mathcal{G}_t] \right)^{-1}.$$

Assume that the mapping $y \rightarrow \mathbb{E}[U(X_{\pi+y\beta}(T))]$ is concave for all bounded controls $\beta \in \mathcal{A}_\mathbb{G}$. Then we also have the converse conclusions

- iii) If M_π is a martingale under $(\mathbb{Q}_\pi, \mathbb{G})$, then M_π is a local maximum.
- iv) If the stochastic process

$$\hat{M}_\pi(t) = M_\pi(t) - \int_0^t \frac{1}{Z(s)} d[M_\pi, Z](s), \quad t \in [0, T],$$

is a martingale under (\mathbb{P}, \mathbb{G}) , then M_π is a local maximum.

Proof.

Part i), if π is a local maximum, then M_π is \mathbb{G} -adapted and has the martingale property by Theorem 3.3.

Part ii) is obtained by application of the Girsanov theorem (see in particular [24, Part III, Theorem 39]).

Part iii) is a direct application of Theorem 3.3.

Part iv) is again an application of the Girsanov Theorem. \square

4. EXAMPLES

We concentrate on the logarithmic utility to reduce computation and highlight some interesting aspects of the analysis. Note that if $U(x) = \ln(x)$ then $F_\pi(T) = 1$ in (3.8). By application of Theorem 3.3 and Lemma 3.5, π is a local maximum if and only if

$$\begin{aligned} 0 &= \mathbb{E} [F_\pi(T) (M_\pi(s) - M_\pi(t)) | \mathcal{G}_t] \\ &= \mathbb{E} [M_\pi(s) - M_\pi(t) | \mathcal{G}_t] \\ &= \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^{s \wedge \tau} [\mu(r) - \rho(r) - \sigma^2(r)\pi(r) - \int_{\mathbb{R}_0} \frac{\pi(r)\theta^2(r, z)}{1 + \pi(r)\theta(r, z)} \nu(dz)] dr \right. \\ &\quad \left. + \int_t^{s \wedge \tau} \sigma(r) d^-W(r) + \int_t^{s \wedge \tau} \int_{\mathbb{R}_0} \frac{\theta(r, z)}{1 + \pi(r)\theta(r, z)} \tilde{N}(dz, d^-r) \right] \end{aligned}$$

$$(4.1) \quad + \int_t^{s \wedge \tau} \frac{\kappa(r)}{1 + \kappa(r)\pi(r)} dH(r) \Big| \mathcal{G}_t \Big].$$

We will consider different cases of \mathbb{G} as forms of partial and anticipating information of the market and default events. The different cases will then be solved using (4.1) and taking limits. This will enable us to see how the information available changes the optimal solution for $\pi(t)$. But first, in order to calculate the expected value of the dH integral, we will need the following lemma and theorem.

Lemma 4.1. *For $s > t$, we have*

$$\mathbb{E} \left[\mathbf{1}_{\{\tau > t\}} (1 - e^{-\int_t^s \lambda(r) dr}) \Big| \mathcal{G}_t \right] = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^{s \wedge \tau} \lambda(r) dr \Big| \mathcal{G}_t \right].$$

Proof. See [5, Section 3.4]. □

Theorem 4.2. *Suppose κ is either \mathcal{G}_t -adapted or independent of λ , then*

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{\{\tau > t\}} \int_t^{s \wedge \tau} \frac{\kappa(r)}{1 + \kappa(r)\pi(r)} dH(r) \Big| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau > t\}} \int_t^s \mathbb{E} \left[\frac{\kappa(r)}{1 + \kappa(r)\pi(r)} \mathbf{1}_{\{\tau > r\}} \lambda(r) \Big| \mathcal{G}_t \right] dr. \end{aligned}$$

Proof. First, note that from (3.1) and (1.9), we have

$$(4.2) \quad \frac{|\kappa(t)|}{1 + \kappa(t)\pi(t)} < \frac{|\kappa(t)|}{\epsilon_\pi} < C < \infty,$$

where C is some constant depending on ϵ_π and the bounds of κ . This allows the following computations. Recall that default events are \mathcal{G}_t -measurable, then

$$\begin{aligned} I(t, s) &:= \mathbb{E} \left[\mathbf{1}_{\{\tau > t\}} \int_t^{s \wedge \tau} \frac{\kappa(r)}{1 + \kappa(r)\pi(r)} dH(r) \Big| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau > t\}} \int_t^{s \wedge \tau} \mathbb{E} \left[\frac{\kappa(r)}{1 + \kappa(r)\pi(r)} \Big| \mathcal{G}_t \right] dH(r). \end{aligned}$$

Writing the integral as the limit of elementary functions;

$$I(t, s) = \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbb{E} \left[\frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \mathbf{1}_{\{\tau \in (t_i, t_{i+1}]\}} \Big| \mathcal{G}_t \right]$$

$$= \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbb{E} \left[\mathbb{E} \left[\frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \mathbf{1}_{\{\tau \in (t_i, t_{i+1}]\}} \middle| \mathcal{G}_{t_i} \right] \middle| \mathcal{G}_t \right],$$

where $t = t_0 < t_1 < \dots < t_N = s$ is a partition of $[t, s]$, and the limit is taken over partitions such that $\Delta t := \sup_i (t_{i+1} - t_i) \rightarrow 0$. We move κ outside of the inner expectation either by measurability or independence, while $\pi(t)$ is \mathcal{G}_t -measurable by definition and we obtain

$$\begin{aligned} I(t, s) &= \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbb{E} \left[\frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \mathbb{E} [\mathbf{1}_{\{\tau \in (t_i, t_{i+1}]\}} \middle| \mathcal{G}_{t_i}] \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbb{E} \left[\frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \mathbb{E} [1 - e^{-\int_{t_i}^{t_{i+1}} \lambda(s) ds} \middle| \mathcal{G}_{t_i}] \middle| \mathcal{G}_t \right]. \end{aligned}$$

By application of Lemma 4.1, we have

$$I(t, s) = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \int_{t_i}^{t_{i+1}} \mathbf{1}_{\{\tau > s\}} \lambda(s) ds \middle| \mathcal{G}_t \right].$$

We want to combine the sum into one integral. Writing the integral as a limit of simple sums, we get two limits of two sums inside each other, which can be merged into one sum. Hence we get

$$\begin{aligned} I(t, s) &= \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\lim_{\Delta t \rightarrow 0} \sum_i \frac{\kappa(t_i)}{1 + \kappa(t_i)\pi(t_i)} \mathbf{1}_{\{\tau > t_i\}} \lambda(t_i) (t_{i+1} - t_i) \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau > t\}} \int_t^s \mathbb{E} \left[\frac{\kappa(r)}{1 + \kappa(r)\pi(r)} \mathbf{1}_{\{\tau > r\}} \lambda(r) \middle| \mathcal{G}_t \right] dr. \end{aligned}$$

□

4.1. Partial information. In the case of partial information, i.e. when there exists some subfiltration \mathcal{E}_t , $t \geq 0$, such that $\mathcal{E}_t \subseteq \mathcal{F}_t \vee \mathcal{H}_t^\Lambda$ and $\mathcal{G}_t = \mathcal{E}_t \vee \mathcal{H}_t^\tau$, the expectation of the forward integrals are zero, so (4.1) can be written as

$$\begin{aligned} 0 &= \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^{s \wedge \tau} \left[\mu(r) - \rho(r) - \sigma^2(r)\pi(r) - \int_{\mathbb{R}_0} \frac{\pi(r)\theta^2(r, z)}{1 + \pi(r)\theta(r, z)} \nu(dz) \right] dr \right. \\ &\quad \left. + \int_t^{s \wedge \tau} \frac{\kappa(r)}{1 + \kappa(r)\pi(r)} dH(r) \middle| \mathcal{G}_t \right]. \end{aligned}$$

Using Theorem 4.2, dividing by $(s - t)$ and letting $s \rightarrow t$, we find that the locally optimal $\pi(t)$ in this case must satisfy

$$0 = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\mu(t) - \rho(t) - \sigma^2(t)\pi(t) \right]$$

$$(4.3) \quad - \int_{\mathbb{R}_0} \frac{\pi(r)\theta^2(r, z)}{1 + \pi(r)\theta(r, z)} \nu(dz) + \frac{\kappa(t)}{1 + \kappa(t)\pi(t)} \lambda(t) \Big| \mathcal{G}_t \Big].$$

If we assume $\theta = 0$, the explicit solution of (4.3) is given by

$$(4.4) \quad \pi = \frac{1}{2\kappa} \left(\frac{\kappa(\hat{\mu} - \hat{\rho})}{\hat{\sigma}^2} - 1 + \sqrt{\left(1 - \frac{\hat{\kappa}(\hat{\mu} - \hat{\rho})}{\hat{\sigma}^2}\right)^2 + 4\hat{\kappa}\left(\frac{\hat{\mu} - \hat{\rho} + \hat{\lambda}\hat{\kappa}}{\hat{\sigma}^2}\right)} \right),$$

where we have set $\hat{\mu}(t) = \mathbb{E}[\mu(t)|\mathcal{G}_t]$ and $\hat{\sigma}^2(s) = \mathbb{E}[\sigma^2(t)|\mathcal{G}_t]$ and so on for the other coefficients.

4.2. Full information. Assume $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^\Lambda \vee \mathcal{H}_t^\tau$. If the coefficients μ , σ , θ , κ and λ are adapted to \mathcal{G}_t , equation (4.3) reduces to

$$(4.5) \quad 0 = \mathbf{1}_{\{\tau > t\}} \left(\mu(t) - \rho(t) - \sigma^2(t)\pi(t) - \int_{\mathbb{R}_0} \frac{\pi(r)\theta^2(r, z)}{1 + \pi(r)\theta(r, z)} \nu(dz) + \frac{\kappa(t)}{1 + \kappa(t)\pi(t)} \lambda(t) \right)$$

In the case of $\theta = 0$, the solution would have the shape (4.4), but with $\hat{\mu}(t) = \mu(t)$, $\hat{\sigma}^2(t) = \sigma^2(t)$ and so on. Remark that for $\lambda = 0$ and $\theta = 0$, equation (4.5) gives us Merton's ratio.

Two explicit examples with full information can be found in Figures 1b and 1a. In Figure 1a the stock price is modeled as

$$(4.6) \quad dS_1(t) = S(t-) \left(\mu_o dt + \sigma dW(t) + \kappa dH(t) \right),$$

with μ_o, σ, κ fixed, and $\rho = 0$. We see that with higher default risk the agent invests less and the asset is also shorted when the overall return becomes negative at the point $\lambda\kappa = -\mu_o$.

In Figure 1b the stock price is modeled as

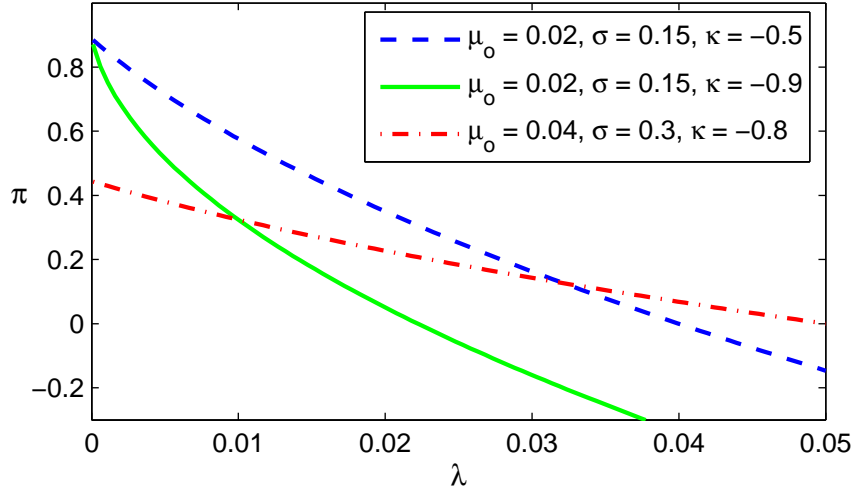
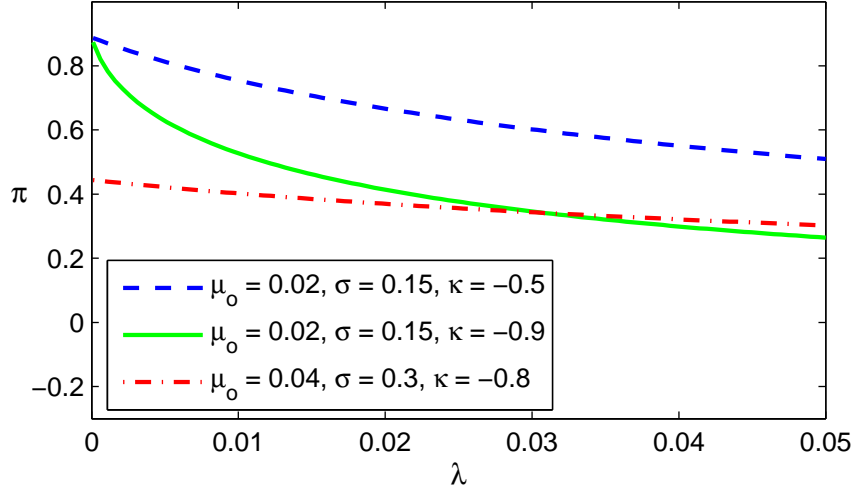
$$(4.7) \quad dS_1(t) = S(t-) \left((\mu_o - \lambda\kappa) dt + \sigma dW(t) + \kappa dH(t) \right)$$

The assumptions in (4.7) are similar to (4.6). But with the term $-\lambda\kappa$ in the drift, the expected return of the asset is invariant to the value of λ . So the agent invests less due to risk aversion and not due to changes in the asset returns.

4.3. Anticipating information. Forward integration allows us to consider anticipating information, i.e. assume that there is a filtration \mathcal{E}_t , $t \geq 0$ with $\mathcal{F}_t \vee \mathcal{H}_t^\Lambda \subset \mathcal{E}_t$, and $\mathcal{G}_t = \mathcal{E}_t \vee \mathcal{H}_t^\tau$.

The cases of anticipating information are more subtle than partial or full information, with various approaches being possible depending on the specific conditions chosen. The challenge with anticipating information is to evaluate the terms $\mathbb{E}\left[\int_t^{s \wedge \tau} \sigma(r) d^-W(r) \Big| \mathcal{G}_t\right]$, $\mathbb{E}\left[\int_t^{s \wedge \tau} \int_{\mathbb{R}_0} \frac{\theta(r, z)}{1 + \pi(r)\theta(r, z)} \tilde{N}(dz, d^-r) \Big| \mathcal{G}_t\right]$ and $\mathbb{E}\left[\int_t^{s \wedge \tau} \frac{\kappa(r)}{1 + \kappa(r)\pi(r)} dH(r) \Big| \mathcal{G}_t\right]$ in (4.1).

One possible way to compute the expectations of the forward integrals above is to exploit Malliavin calculus, see [14, Chapter 8 and Chapter 15] for the theoretical framework.

(A) Default risk *not* compensated by higher drift

(B) Default risk compensated by higher drift

FIGURE 1. Optimal investment π as a function of λ .

However we must stress that this general approach cannot always be taken here. In fact the application of Malliavin calculus requires that the integrands are measurable with respect to \mathcal{F}_T , which is not, in general, the case when considering default risk.

See also [13] on how the $\tilde{N}(d^-t, dz)$ integral can be evaluated using predictable compensators of the measure with respect to \mathbb{G} and [2, 22, 20] for other examples on the d^-W integral in insider models without default risk.

In the coming example we consider explicitly the presence of default risk. Since τ , the jump of H , is a totally inaccessible stopping time, the most natural way to model insider knowlegde is by allowing different information on the λ process.

4.4. Example: Honest and insider optimal trading compared. We introduce a model for the stochastic intensity to exemplify how inside information comes into play at decision making level, especially compared to the trader who has not access to such information.

Assume $\nu(dz) = \gamma \mathbf{1}_{\{b\}}(z)$, i.e. the Poisson random measure $N(dt, dz)$ corresponds to Poisson process $N(t)$, $t \geq 0$ with intensity γ and fixed jump sizes b . Assume $\delta > 0$ and let λ be given by a non-Gaussian Ornstein-Uhlenbeck process (see for instance [8, section 15.3]) of the form

$$\lambda(t) = \lambda_0 e^{-at} + \int_{\delta}^{t+\delta} e^{a(s-t)} N(ds).$$

We let the stock price dynamics, (1.2), be given by

$$dS_1(t) = S_1(t-) \left([\mu - \lambda(t - \delta)\kappa] dt + \sigma dW(t) + \theta d\tilde{N}(t) + \kappa dH \right),$$

so both the default intensity and stock price are affected by the jumps of N , but with a time difference δ . For simplicity μ , δ , κ , σ and θ are assumed to be constants.

Loosely, the idea is that λ denotes the financial health of the firm. The parameter δ is a delay between increased default risk and when the market is made aware of the increase. The term $\theta d\tilde{N}(t)$ then influences how the market reacts to these news, with an upward or downward jump. The term $-\lambda(t - \delta)\kappa dt$ means that, as far as the market is aware, the asset return reflects the current level of default risk.

The honest investor. Assume $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^\tau$. From (4.3) we see that the optimal π is a solution of

$$0 = \mu - \lambda(t - \delta)\kappa - \sigma^2 \pi(t) - \frac{\pi(t)\theta^2}{1 + \pi(t)\theta} b\gamma + \frac{\kappa}{1 + \pi(t)\kappa} \mathbb{E}[\lambda(t) | \lambda(t - \delta)]$$

with

$$\begin{aligned} \mathbb{E}[\lambda(s) | \lambda(t)] &= \lambda(t) e^{-a(s-t)} + \frac{\mathbb{E}[N(1)]}{a} (1 - e^{-a(s-t)}) \\ &= \lambda(t) e^{-a(s-t)} + \frac{b\gamma}{a} (1 - e^{-a(s-t)}). \end{aligned}$$

The insider. Now assume that an investor knows roughly about the number of bad news arriving, by assuming $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^\tau \vee \sigma(N(T_0))$ for some $T_0 > T$. Starting from (4.1), we get a criteria for the optimal π in this case, which we will reduce to a simpler one using Theorem 4.2, Lemma 2.2 and the forthcoming Lemma 4.3. Hence we have

$$0 = \mathbb{E} \left[\int_t^{s \wedge \tau} \left(\mu - \lambda(r - \delta)\kappa - \sigma^2 \pi(r) - \frac{\pi(r)\theta^2}{1 + \pi(r)\theta} \right) dr \right]$$

$$\begin{aligned}
& - \int_t^{s \wedge \tau} \sigma d^- W(r) - \int_t^{s \wedge \tau} \frac{\theta}{1 + \pi(r)\theta} d^- N(r) + \int_t^{s \wedge \tau} \frac{\kappa}{1 + \pi(r)\kappa} dH(r) \Big| \mathcal{G}_t \Big] \\
0 = & \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^{s \wedge \tau} \left(\mu - \lambda(r - \delta)\kappa - \sigma^2 \pi(r) - \frac{\pi(r)\theta^2}{1 + \pi(r)\theta} \right. \right. \\
& \left. \left. - \frac{\theta}{1 + \pi(r)\theta} \frac{N(T_0) - N(r)}{T_0 - r} + \frac{\kappa}{1 + \pi(r)\kappa} \lambda(r) \right) dr \Big| \mathcal{G}_t \right]
\end{aligned}$$

Dividing on both sides by $(s - t)$ and taking limits, we deduce that the optimal control for the insider is a solution of

$$\begin{aligned}
0 = & \mu - \lambda(t - \delta)\kappa - \sigma^2 \pi(t) - \frac{\pi(t)\theta^2 + \theta}{1 + \pi(t)\theta} b\gamma \\
& + \frac{\theta}{1 + \pi(t)\theta} \frac{N(T_0) - N(t)}{T_0 - t} + \frac{\kappa}{1 + \pi(t)\kappa} \mathbb{E}[\lambda(t) | \mathcal{G}_t]
\end{aligned}$$

with $\mathbb{E}[\lambda(t) | \mathcal{G}_t]$ given by (4.9).

Lemma 4.3. *If $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^\tau \vee \sigma(N(T_0))$ then*

i)

$$(4.8) \quad \mathbb{E} \left[\int_t^{s \wedge \tau} \frac{\theta}{1 + \pi(r)\theta} N(d^- r) \Big| \mathcal{G}_t \right] = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^{s \wedge \tau} \frac{\theta}{1 + \pi(r)\theta} \frac{N(T_0) - N(r)}{T_0 - r} dr \Big| \mathcal{G}_t \right].$$

ii)

$$\begin{aligned}
(4.9) \quad \mathbb{E}[\lambda(t) | \mathcal{G}_t] &= \lambda(t - \delta)e^{-a\delta} + \frac{N(T_0) - N(t)}{T_0 - t} \int_t^{t+\delta} e^{a(s-t)} ds \\
&= \lambda(t - \delta)e^{-a\delta} + \frac{N(T_0) - N(t)}{T_0 - t} \frac{e^{a\delta} - 1}{a}.
\end{aligned}$$

Proof.

i) Note that conditional on $N(T_0)$, the jump times of N are uniformly distributed on $[0, T_0]$, see for instance [9, Chapter 2.2], so that

$$\mathbb{E}[N(s) - N(t) | N(T_0), N(t)] = \frac{s - t}{T_0 - t} (N(T_0) - N(t)) \quad \text{for } t < s \leq T_0$$

This gives

$$\mathbb{E} \left[\int_t^{s \wedge \tau} \frac{\theta}{1 + \pi(r)\theta} N(d^- r) \Big| \mathcal{G}_t \right]$$

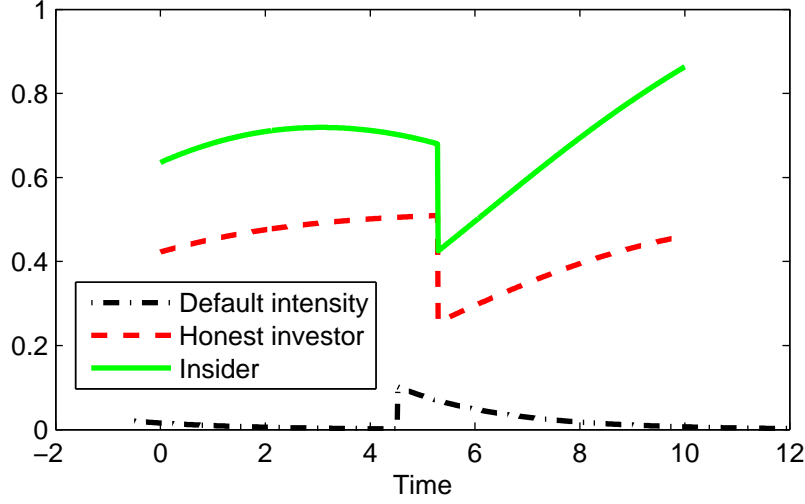


FIGURE 2. Insider, honest investor and default intensity, example 1

$$\begin{aligned}
(4.10) \quad &= \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbf{1}_{\{\tau > t_i\}} \frac{\theta}{1 + \pi(t_i)\theta} (N(t_{i+1}) - N(t_i)) \middle| \mathcal{G}_t \right] \\
&= \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbf{1}_{\{\tau > t_i\}} \frac{\theta}{1 + \pi(t_i)\theta} \mathbb{E} \left[\mathbb{E} \left[(N(t_{i+1}) - N(t_i)) \middle| \mathcal{G}_t \vee \sigma(N(t_i)) \right] \middle| \mathcal{G}_t \right] \\
&= \mathbf{1}_{\{\tau > t\}} \lim_{\Delta t \rightarrow 0} \sum_{i=0}^{N-1} \mathbf{1}_{\{\tau > t_i\}} \frac{\theta}{1 + \pi(t_i)\theta} \mathbb{E} \left[\frac{t_{i+1} - t_i}{T_0 - t_i} (N(T_0) - N(t_i)) \middle| \mathcal{G}_t \right] \\
&= \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\int_t^s \frac{\theta}{1 + \pi(r)\theta} \frac{N(T_0) - N(r)}{T_0 - r} dr \middle| \mathcal{G}_t \right].
\end{aligned}$$

In (4.10), $t = t_0 < t_1 < \dots < t_N = s$ is a partition of $[t, s]$, and the limit is taken over partitions such that $\Delta t := \sup_i (t_{i+1} - t_i) \rightarrow 0$.

ii) The calculations are similar to i).

□

An example on how the honest investor and insider would invest differently can be found in Figures 2 and 3. A path for $N(t)$ was simulated, then the optimal investment calculated for both the insider and the honest investor. When default risk increases, both investors decrease their holdings due to risk aversion. But the insider, using the information of $N(T_0)$ invests more or less aggressively depending on the number of remaining jumps of N . The parameters for the figures were $\lambda_0 = 0.02$, $a = 0.5$, $b = 0.1$, $\delta = 0.5$, $\mu = 0.03$, $\sigma = 0.2$, $\gamma = 0.3$, $\kappa = -0.8$, $\theta = -0.3$, $T_0 = 12$, $T = 10$.

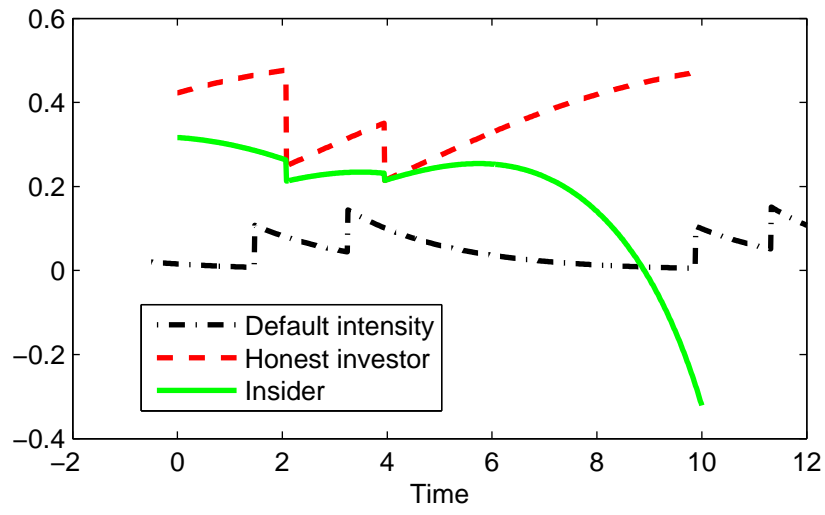


FIGURE 3. Insider, honest investor and default intensity, example 2

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